Langevin description of the response of a stochastic mean-field model driven by a time-periodic field

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We analyze the dynamical response of a nonlinear stochastic model with mean-field coupling driven by a time-sinusoidal external field. The Langevin equation for the model is solved numerically, and the results indicate the possibility of observing stochastic resonant amplification of the driving amplitude. The influence of the mean-field coupling on the typical stochastic resonance effects is pointed out.

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The subject of amplification of an external signal by stochastic systems has been an active field of research in recent years [1]. In particular, within the context of stochastic resonance, a great deal of work has been devoted to the analysis of the response of a stochastic symmetric bistable system in the limit of very large damping. In this work we are interested in the study of the stochastic amplification in a driven system that exhibits, in the thermodynamic limit, an order-disorder phase transition. The model consists of very many subunits with mean-field interactions between them. It was introduced by Kometani and Shimizu [2] within the context of muscle contraction, and a more statistical mechanical treatment was later given by Desai and Zwanzig [3] and by Dawson [4] which pointed out its relation with the Weiss-Ising model.

In the limit of a very large number of subunits, and in the presence of a driving force, the order parameter x satisfies the Langevin equation

$$\dot{x}(t) = (1-\theta)x(t) - x^3(t) + A\cos\Omega t + \theta\langle x(t)\rangle + \xi(t), \tag{1}$$

where $\xi(t)$ is a white Gaussian noise with zero mean and $\langle\xi(t)\xi(s)\rangle = 2D\delta(t-s)$, θ represents the strength of the mean-field coupling among the subunits, and $A\cos\Omega t$ represents the effect of the driving field. $\langle x(t)\rangle$ is the average of $x(t)$. This Langevin equation can be thought of as describing the Brownian motion of a particle in an effective potential U_{eff},

$$U_{eff}(x,\langle x(t)\rangle,t) = (\theta-1)\frac{x^2}{2} + \frac{x^4}{4} - \theta\langle x(t)\rangle x - Ax\cos\Omega t, \tag{2}$$

which depends upon the state of the system through the average $\langle x(t)\rangle$. The effect of the mean-field coupling is twofold: it changes the curvature of the extrema of the function U_{eff} and it might render it asymmetric. One can also look at U_{eff} as a symmetric double well with a barrier height which depends upon θ, and is subject to a time-dependent perturbation which makes it asymmetric.

When $\theta = 0$, i.e.,when the mean-field interaction is absent, each subunit evolves in time separately, according to the Langevin equation (1) with $\theta = 0$. In this case, one has a situation which has been repeatedly studied by many authors as an archetypical case for stochastic resonance. It should be pointed out that even though the Langevin equation is still nonlinear, the corresponding Fokker-Planck equation (FPE) is linear in the probability density. A perturbative analysis of this linear FPE for a driven system, based on eigenfunction expansions and the Floquet theory, was given in Ref. [5], where it is shown that the system has the mixing property and the long-time solution, which is time dependent, is always reached regardless of the initial condition.

When $\theta \neq 0$, the FPE corresponding to the Langevin description is nonlinear in the probability density and the system presents an order-disorder phase transition. In the absence of the driving field ($A = 0$), and for each value of θ, there exists a value of the noise strength D_c, so that for $D > D_c$ there is just one stable equilibrium distribution function for the stochastic variable x, with $\langle x\rangle_{eq} = 0$, while for $D < D_c$ there are two stable equilibrium distributions with $\langle x\rangle_{eq} = \mp x_0$ with x_0 depending on θ and D. Thus, at the critical line, there is a bifurcation of the equilibrium probablity density. For $D < D_c$ the equilibrium distribution is always single peaked, while for $D > D_c$, the stable equilibrium distribution has either two or one maxima depending on whether θ is less than or larger than 1. The phase diagram is sketched in Fig. 1 in terms of the reduced variable $|z| = |\theta - 1|(2D)^{-\frac{1}{2}}$.

A few years ago, Shiino was able to prove an H theorem for the nonlinear Fokker-Planck equation for the undriven system [6]. Therefore, in the long-time limit, the system always reaches stationary situations characterized by time-independent equilibrium distributions. Clearly, for a given θ and $D > D_c$, the equilibrium situation is unique, regardless of the initial condition. On the other hand, for $D < D_c$, there are two stable equilibrium solutions and, in the long-time limit, the system reaches one or the other depending upon the initial preparation of the system. In this sense, we can say that the mean-field interaction breaks the ergodicity of the process.

When a time-dependent field is present, we have not been able to extend Shiino's H theorem. On the other hand, the nonlinearity of the Fokker-Planck dynamics

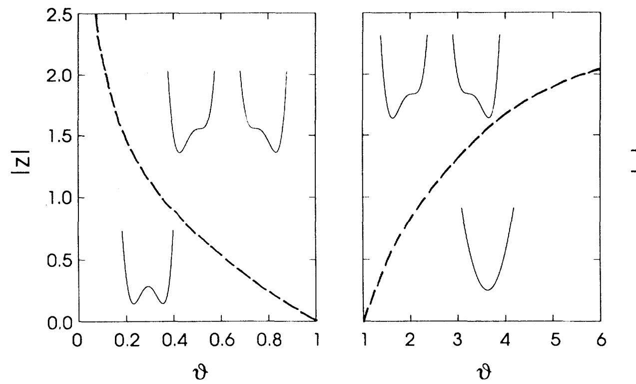

FIG. 1. Equilibrium phase diagram for the model. The dashed line is the critical line. The insets are sketches of the different U_{eff} for $\langle x\rangle = \langle x\rangle_{eq}$.

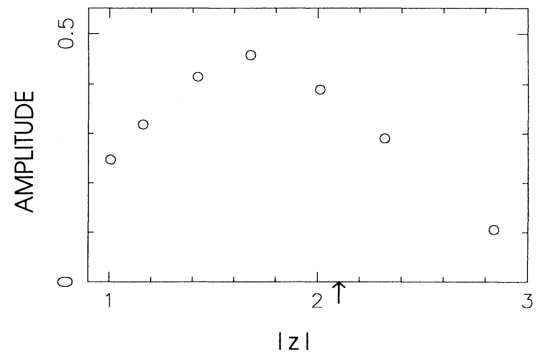

FIG. 2. Amplitude of $\langle x(t)\rangle$ as a function of $|z|$ for $A = 0.1, \theta = 0.1$, and $\Omega = 0.1$. The arrow marks the value of $|z_c|$.

prevents us from making use of the Floquet theory. For very weak driving amplitudes, the long-time behavior of the system can be understood by using first order perturbation theory. In this limit, the system describes long-time oscillations about the corresponding equilibrium values. The external field is not able to restore the ergodicity of the process and there are still two disconnected distribution functions for points above the critical line. A detailed analysis of the linear response of the system, based on a perturbative analysis of the corresponding nonlinear Fokker-Planck equation, will be carried out elsewhere [7]. In this paper, in order to analyze the response of the system to a driving field (not necessarily very weak), we have resorted to the numerical solution of the Langevin equation, by generating a sufficiently large number of stochastic trajectories (5000 in most cases) and averaging over them. This technique was previously used by us in an analysis of the model in the absence of the external field and the details can be found in Ref. [8]. We will restrict our simulation to $\theta < 1$, as this is the region for which bimodal distribution functions exist and the phenomenon of stochastic resonance is expected.

Let us first consider the results of the numerical simulation for $A = 0.1$, $\Omega = 0.1$, and $\theta = 0.1$. The noise average $\langle x(t)\rangle$ shows oscillatory behavior for long times. Away from the critical line the centers of the oscillations are the corresponding equilibrium values, while near the critical line, they are slightly shifted with respect to $\langle x\rangle_{eq}$. In Fig. 2 the amplitude of the oscillations is plotted versus $|z|$. Also the critical value $|z_c|$ is indicated. For $|z| > |z_c|$ (i.e., D rather small), the amplitude of the oscillations is roughly the same as the amplitude of the driving field. The system is in a region where the distribution has a single maximum and, thus, one should expect that the external driving simply induces small oscillations about the minimum of the well. On the other hand, as soon as $|z|$ gets smaller than $|z_c|$ (i.e., D increases), the effective potential affecting the order parameter has two wells. Noise can induce transitions among these wells and we have a possibility of amplification of the signal, as the results indicate. The maximum amplification takes place at $D \approx 0.14$ ($|z| \approx 1.7$). For this value of the noise, we have that twice the Kramers frequency as given by

$$2\omega_{\mathrm{Kr}} = \frac{\sqrt{2}}{\pi}(1-\theta)\exp\left(-\frac{z^2}{2}\right) \quad (3)$$

is approximately 0.095, a value not too different from the external frequency. This indicates that for small values of θ, the stochastic resonance mechanism in a mean-field model is not too different from the usual one in the absence of mean-field coupling.

In Fig. 3 we show the amplitude of the oscillations about the corresponding equilibrium values for a larger strength of the mean-field coupling, $\theta = 0.5$, and for external field parameters $A = 0.05$ and $\Omega = 0.35$. The amplitude of the driving field has been reduced with respect to its value in Fig. 2, due to the fact that, as θ increases, the height of the barrier of the symmetric double well is decreased. Thus, in order to analyze the response to a weak field, A must be reduced accordingly. We notice that there is still an enhancement of the response with respect to the input signal, but this enhancement is now smaller than in the previous case, due to the larger value of the driving frequency. The maximum amplification takes place at $D = 0.3$, a value which is rather large for the Kramers formula, given by Eq. (2), to apply. Our results indicate that, as θ is increased, the hopping mechanism between wells is influenced by the mean-field interaction.

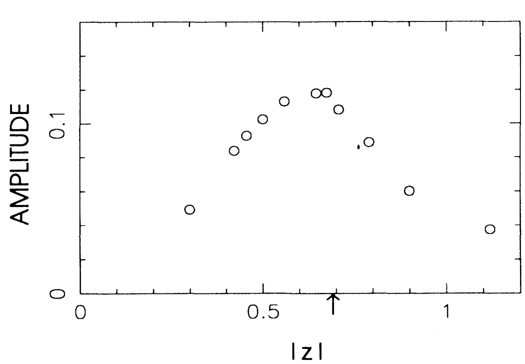

FIG. 3. Amplitude of $\langle x(t)\rangle$ as a function of $|z|$ for $A = 0.05, \theta = 0.5$, and $\Omega = 0.35$. The arrow marks the value of $|z_c|$.

Next we analyze the response of the system to a rather strong driving force. In Fig. 4, we show the behavior of the amplitude of the response with respect to $|z|$ for $\theta = 0.5$, $\Omega = 0.1$, and A=0.2. For $|z|$ greater than $|z_c|$, the interesting feature is that the system's response is not really periodic in time for $|z_c| < |z| < 1.2$. In the absence of external driving there exist two distinct stable distribution functions in this range of parameters. The external driving seems to connect them in such a way that the system switches between them. One could think that a strong enough external field would be able to restore the ergodicity of the process. But this is not the case as, for sufficiently large $|z|$ ($|z| > 1.2$), the response of the system is again periodic in time, with oscillations around one of the two stable equilibrium average values. For these values of $|z|$, one can still consider that the two different distributions exist. Thus, we believe that what happens is that the position of the critical line depends upon the value of A and it is shifted noticeably with respect to the undriven case for large external driving forces.

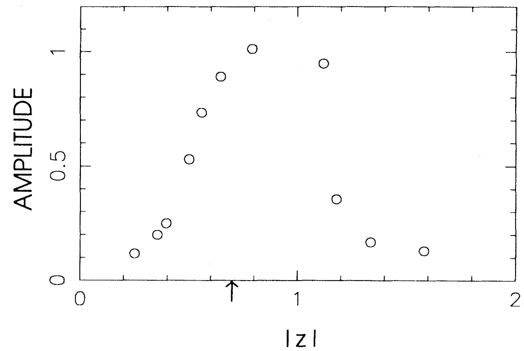

FIG. 4. Amplitude of $\langle x(t) \rangle$ as a function of $|z|$ for $A = 0.2, \theta = 0.5$, and $\Omega = 0.1$. The arrow marks the value of $|z_c|$.

In conclusion, our numerical study of the Langevin equation of the model shows that the phenomenon of noise amplification of a weak signal is still present in the mean-field model. For small θ, the mechanism of amplification is similar to the one leading to stochastic resonance in the usual bistable model ($\theta = 0$). On the other hand, as θ is increased, the hopping mechanism between wells is influenced by the mean-field coupling.

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